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Vibration analysis of flexural-shear plates with varying cross-section

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Abstract

In this paper, multi-storey buildings with narrow rectangular plane configuration (narrow buildings) are treated as cantilever flexural-shear plates in analysis of free vibration. The governing differential equations for free vibration of flexural-shear plates with variably distributed mass and stiffness are established and reduced to Bessel's equations or Euler's equation by selecting suitable expressions, such as power functions and exponential functions, for the distributions of stiffness and mass along the height of the plates. The general solutions of flexural-shear plates are derived. Numerical examples demonstrate that the calculated natural frequencies and mode shapes of narrow buildings are in good agreement with the experimentally measured data. It is also shown that it is possible to regard a building with rigid floors as a cantilever flexural bar that is a special case of a cantilever flexural-shear plate. Thus, the methods proposed in this paper are suitable for the calculation of free vibration of narrow buildings and common shear-wall buildings.

Keywords: Vibration; Tall buildings; Plates; Natural frequencies; Mode shapes

1. Introduction

The full-scale measurements of free vibration of buildings (e.g., Wang, 1978; Li et al., 1994; Jeary, 1997) have shown that the flexural deformation is dominant in the total deformation of multi-storey buildings with shear-wall structures in their horizontal vibrations. Li et al. (1996) suggested that for certain cases these shear-wall buildings can be simplified as cantilever flexural bars for the analysis of their free vibrations. An approach to determine the natural frequencies and mode shapes of cantilever flexural bars with variably distributed mass and stiffness was proposed by Li (1995). However, if a building has a narrow rectangular plane configuration, $B/L < 1/4$, where B and L are the width and

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length of the rectangular plane, respectively, the stiffness of each floor of the building cannot be treated as infinitely rigid. Then, this building may not be simplified as a cantilever bar for free vibration analysis. It is reasonable to consider such a building as a cantilever plate. The field measurements conducted by Ishizaki and Katakeyana (1960), Wang (1978), Li et al. (1994) and Jeary (1997) revealed that for multi-storey buildings with a narrow rectangular plane configuration (narrow building), shear deformation is dominant in the total deformation in their horizontal vibrations for certain cases. They reported that not only the parallel motions between floors occurred, but also the relative motions between parallel frames were observed. Thus, when analysing free vibration of narrow buildings, it is possible to regard such structures as cantilever shear plates with variably distributed mass and stiffness (Wang, 1978). However, a narrow building with shear-wall structures may not be simplified as a cantilever shear plate. This is due to the fact that it has been found (e.g., Li et al., 1994; Jeary, 1997) that the flexural deformation of shear-walls is dominant in the total deformation in the horizontal vibrations of such a narrow building. Hence, it is more resonable to treat a narrow building with shear-wall structures as a cantilever flexural-shear plate with variably distributed mass and stiffness.

In fact, there are very few equations of vibrating plates with variable cross-section where exact solutions can be obtained. These exact plate solutions are available only for certain plate shapes and boundary conditions (e.g., Timoshenko and Woinowsky-Krieger, 1959). Chopra (1974) developed an analytical approach for the free vibration of a simply supported plate with one change in thickness. Guo et al. (1997) recently found the analytical solutions for the free vibration of a stepped, simply supported plate with uniform thickness and abrupt thickness changes. Wang (1978) derived the closed form solutions for the free vibration of cantilever shear plates with uniformly distributed mass and stiffness. However, it is obvious that the distributions of mass and shear stiffness of most narrow buildings are actually not uniform, especially, along the building height. In general, the variation of mass and stiffness along the longitudinal axis of a narrow building (the x-axis in Fig. 1a) can be neglected. Thus, it is reasonably assumed that the narrow buildings considered in this paper have uniformly distributed mass and stiffness along the longitudinal axis, but variably distributed mass and stiffness along the height of the narrow buildings. Free vibration analysis of a cantilever flexural-shear plate with variably distributed

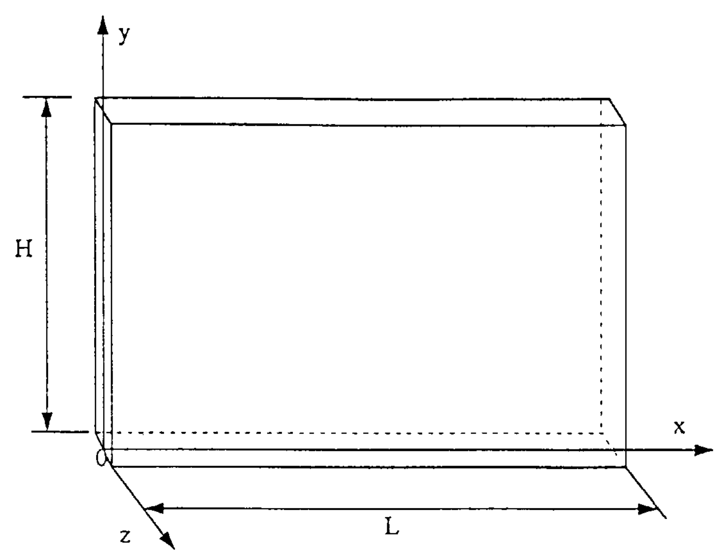

Fig. 1a. A one-step flexural-shear plate.

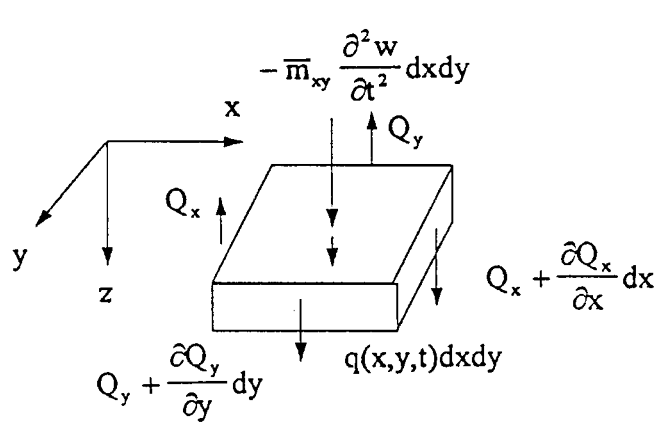

Fig. 1b. An element of the plate.

mass and stiffness has received relatively little attention in the past. The exact solution of this problem has not previously been proposed in the literature. In this paper, the distributions of mass and stiffness along the height of the plates are described by selecting suitable functions, such as power functions and exponential functions. The exact solutions of flexural-shear plates with variably distributed mass and stiffness are derived. The numerical examples show that the calculated dynamic characteristics of two tall buildings are in good agreement with the measured field data. It is shown through the numerical examples that the selected expressions are suitable for describing the distributions of mass and stiffness of typical multi-storey buildings.

In this paper, an attempt is made to present exact analytical solutions to free vibrations of cantilever flexural-shear plates with variably distributed mass and stiffness. In the absence of exact solutions, this problem can be solved using approximated methods (e.g., the Ritz method) or numerical methods (e.g., the finite element method and the finite strip method). However, the present exact solutions can provide adequate insight into the physics of the problem and can be easily implemented. The availability of the exact solutions will help in examining the accuracy of the approximate or numerical solutions. Therefore, it is always desirable to obtain the exact solutions to such problems.

2. The governing differential equation

A narrow building is simplified as a cantilever flexural-shear plate as shown in Fig. 1a. Its deformation in the lateral direction is flexural, but the deformation in the longitudinal direction (the x-axis in Fig. 1a) is shear. It is assumed that this plate is subjected to a transversal dynamic load, $q(x, y, t)$. In order to establish the governing differential equation of vibration of this plate, an infinitesimal element of the plate is taken, as shown in Fig. 1b. The size of the element is $dx \times dy$. The dynamic loading acting on the element is $q(x, y, t)\,dx\,dy$. The inertial force is $[-\bar{m}_{xy}(\partial^2 w/\partial t^2)\,dx\,dy]$, where w is the dynamic displacement of the plate in the z-axis at the point (x, y). The damping force is $[-C_{xy}(\partial w/\partial t)\,dx\,dy]$. Fig. 1b shows the element that is rotated through an angle of 90°. Considering the equilibrium conditions for all the forces acting on the element (Fig. 1b), using d' Alembert's principle, leads to:

$$\left[\left(Q_y + \frac{\partial Q_y}{\partial y}\,\mathrm{d}y\right) - Q_y\right]\mathrm{d}x + \left[\left(Q_x + \frac{\partial Q_x}{\partial x}\,\mathrm{d}x\right) - Q_x\right]\mathrm{d}y + q(x, y, t)\,\mathrm{d}x\,\mathrm{d}y$$
$$- C_{xy}\frac{\partial w}{\partial t}\,\mathrm{d}x\,\mathrm{d}y - \bar{m}_{xy}\frac{\partial^2 w}{\partial t^2}\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{1}$$

Thus,

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - C_{xy}\frac{\partial w}{\partial t} - \bar{m}_{xy}\frac{\partial^2 w}{\partial t^2} = -q(x, y, t) \tag{2}$$

in which $\bar{m}_{xy}$ and C_{xy} are the mass intensity (mass per unit area) and viscous damping coefficient at the point (x, y), respectively.

Because the deformation in the x-axis is shear and that in the y-axis is flexural, we have,

$$Q_y = \frac{\partial M_y}{\partial y} = -\frac{\partial}{\partial y}\left(K_y\frac{\partial^2 w}{\partial y^2}\right) \quad Q_x = K_x\frac{\partial w}{\partial x} \tag{3}$$

where K_x and K_y are the transverse shear stiffness in the x-axis and transverse flexural stiffness in the y-axis, respectively. M_y is the bending moment about the x-axis.

Substituting eqn (3) into eqn (2) leads to

$$\frac{\partial}{\partial y}\left(K_x\frac{\partial w}{\partial x}\right) - \frac{\partial^2}{\partial y^2}\left(K_y\frac{\partial^2 w}{\partial y^2}\right) - C_{xy}\frac{\partial w}{\partial t} - \bar{m}_{xy}\frac{\partial^2 w}{\partial t^2} = -q(x, y, t) \tag{4}$$

This is the governing differential equation for vibration of a flexural-shear plate with variably distributed mass and stiffness along the height of the plate. Setting $q(x, y, t) = 0$ gives the governing differential equation for free vibration of the flexural-shear plate as follows

$$\frac{\partial}{\partial y}\left(K_x\frac{\partial w}{\partial x}\right) - \frac{\partial^2}{\partial y^2}\left(K_y\frac{\partial^2 w}{\partial y^2}\right) - C_{xy}\frac{\partial w}{\partial t} - \bar{m}_{xy}\frac{\partial^2 w}{\partial t^2} = 0 \tag{5}$$

In general, the damping force is not considered for free vibration analysis, so,

$$\frac{\partial}{\partial y}\left(K_x\frac{\partial w}{\partial x}\right) - \frac{\partial^2}{\partial y^2}\left(K_y\frac{\partial^2 w}{\partial y^2}\right) - \bar{m}_{xy}\frac{\partial^2 w}{\partial t^2} = 0 \tag{6}$$

In order to determine natural frequencies and mode shapes, it is assumed that

$$w(x, y, t) = Z(x, y)\sin(\omega t + \gamma_0) \tag{7}$$

where $Z(x, y)$ is the vibration mode function, ω is the circular natural frequency, γ_0 is the initial phase.

Substituting eqn (7) into eqn (6) gives

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial Z}{\partial x}\right) - \frac{\partial^2}{\partial y^2}\left(K_y\frac{\partial^2 Z}{\partial y^2}\right) + \bar{m}_{xy}\omega^2 Z = 0 \tag{8}$$

In order to simplify the calculation and get the analytical solutions, it is assumed that K_y is a function of y, K_x and $\bar{m}_{xy}$ are also functions of y.

$$K_y = K_1 f(y), \quad K_x = K_2 \varphi(y), \quad \bar{m}_{xy} = \bar{m}\varphi(y) \tag{9}$$

i.e., it is assumed that K_x is directly proportional to $\bar{m}_{xy}$ since the values of K_x and $\bar{m}_{xy}$ are mainly dependent on the size and materials of building floors. In fact, the mass and stiffness distribution of each floor is usually approximately uniform along the x-axis, thus, this assumption is reasonable for most narrow buildings.

Using the method of separation of variables gives

$$Z(x, y) = X(x)Y(y) \tag{10}$$

Substituting eqns (9) and (10) into eqn (8) leads to

$$K_2 \frac{\dfrac{d^2 X}{dx^2}}{X} + \bar{m}\omega^2 = \frac{\dfrac{d^2}{dy^2}\left[K_1 f(y)\dfrac{d^2 Y}{dy^2}\right]}{Y\varphi(y)} \tag{11}$$

The left-hand-side of the above equation is a function of y and the right-hand-side is a function of x. Thus, both sides should be equal to a constant. It is assumed that the constant is $\bar{m}\theta^2$ then, two ordinary differential equations are obtained from eqn (11) as follows

$$K_2 \frac{d^2 X}{dx^2} + \bar{m}\Omega^2 X = 0 \tag{12}$$

$$\frac{d^2}{dy^2}\left[K_1 f(y)\frac{d^2 Y}{dy^2}\right] - \bar{m}\varphi(y)\theta^2 Y = 0 \tag{13}$$

where

$$\Omega^2 = \omega^2 - \theta^2 \quad \omega = \sqrt{\Omega^2 + \theta^2} \tag{14}$$

The boundary conditions of the cantilever flexural-shear plate (Fig. 1a) are as

$$x = 0, \quad Q_x = 0, \quad \text{i.e.,} \quad \left.\frac{dX}{dx}\right|_{x=0} = 0 \tag{15}$$

$$x = L, \quad \left.\frac{dX}{dx}\right|_{x=L} = 0 \tag{16}$$

$$y = 0, \quad Y(0), \quad \left.\frac{dY}{dy}\right|_{y=0} = 0 \tag{17}$$

$$y = H, \quad M_y(H) = 0, \quad Q_y(H) = 0, \quad \text{i.e.,} \quad \left.\frac{d^2 Y}{dy^2}\right|_{x=H} = 0, \quad \left.\frac{d}{dy}\left(K_y \frac{d^2 Y}{dy^2}\right)\right|_{x=H} = 0 \tag{18}$$

It is obvious that eqns (12) and (13) are the governing differential equations of vibration mode shape of a shear bar and a flexural bar, respectively. The boundary conditions of the shear bar are described by eqns (15) and (16). The boundary conditions of the flexural bar are represented by eqns (17) and (18). The natural frequency of the plate is equal to the square root of the square sum of the two natural frequencies of the two bars. The mode shape of the plate is the product of the corresponding two mode shapes of the two bars.

3. The general solutions of free vibration of flexural-shear plates

As discussed above, free vibration analysis can be carried out by analysing two independent bars, i.e., by solving the two independent ordinary differential equations, eqns (12) and (13).

The general solution of eqn (12) is found as

$$X(x) = D_1 \sin\frac{\Omega}{\alpha_2}x + D_2 \cos\frac{\Omega}{\alpha_2}x \tag{19}$$

where

$$\alpha_2 = \sqrt{\frac{K_2}{\bar{m}}} \tag{20}$$

Using the boundary conditions, eqns (15) and (16), gives

$$D_1 = 0 \tag{21}$$

$$\sin\frac{\Omega}{\alpha_2}L = 0 \tag{22}$$

The k-th circular natural frequency and mode shape of the shear bar in the x-axis are as follows

$$\Omega_k = \frac{(k-1)\pi}{L}\sqrt{\frac{K_2}{\bar{m}}} \tag{23}$$

$$X_k(x) = D_2 \cos\frac{(k-1)\pi x}{L} \tag{24}$$

D_2 can be taken as any value, for example, $D_2 = 1$.

It is difficult to find the exact solution of eqn (13) for general cases, because the structural parameters in the equation vary with co-ordinate y. It is obvious that the exact solution is dependent on the distributions of mass and stiffness. The exact solution of eqn (13) may be obtained by means of reasonable selections for the mass and stiffness distributions. As suggested by Wang (1978), Tuma and Cheng, 1983 and Li et al. (1994, 1996, 1998), the functions that can be used to approximate the variation of mass and stiffness are algebraic polynomials, exponential functions, trigonometric series, or their combinations. In this paper, four important cases are considered and discussed as follows.

Case 1:

$$K_1(y) = K_1 = \text{constant} \quad \bar{m}_{xy} = \bar{m} = \text{constant} \tag{25}$$

It is obvious that Case 1 represents a uniform flexural-shear plate.

Substituting eqn (25) into eqn (13) gives

$$\frac{d^4 Y}{dy^2} - \bar{k}^4 Y = 0 \tag{26}$$

where

$$\bar{k} = \sqrt{\frac{\theta}{\alpha_1}}, \quad \alpha_1 = \sqrt{\frac{K_1}{\bar{m}}} \tag{27}$$

The general solution of eqn (26) is found as,

$$Y(y) = C_1 e^{\bar{k}y} + C_2 e^{-\bar{k}y} + C_3 \sin \bar{k}y + C_4 \cos \bar{k}y \tag{28}$$

Case 2:

$$K_1(y) = K_1(1 + \beta y)^{n+2} \tag{29}$$

$$\bar{m}_{xy} = \bar{m}(1 + \beta y)^n \tag{30}$$

eqn (13) can be rewritten as

$$K_y \frac{d^4 Y}{dy^2} + 2\frac{dK_y}{dy}\frac{d^3 Y}{dy^3} + \frac{d^2 K_y}{dy^2}\frac{d^2 Y}{dy^2} - \bar{m}_{xy}\theta^2 Y = 0 \tag{31}$$

Substituting eqns (29) and (30) into eqn (31) gives

$$(1 + \beta y)^2 \frac{d^4 Y}{dy^2} + 2\beta(n+2)(1 + \beta y)\frac{d^3 Y}{dy^3} + \beta^2(n+2)(n+1)\frac{d^2 Y}{dy^2} - \frac{\bar{m}}{K_1}\theta^2 Y = 0 \tag{32}$$

A differential operator, Δ, is introduced herein

$$\Delta = \frac{1}{(1 + \beta y)^n}\frac{d}{dy}\left[(1 + \beta y)^{n+1}\frac{d}{dy}\right] \tag{33}$$

Then, eqn (32) can be written as

$$\left(\Delta + \frac{\omega}{\alpha_1}\right)\left(\Delta - \frac{\omega}{\alpha_1}\right) Y = 0 \tag{34}$$

where

$$\alpha_1 = \sqrt{\frac{K_1}{\bar{m}}} \tag{35}$$

i.e., eqn (34) can be divided into two differential equations

$$\left(\Delta + \frac{\omega}{\alpha_1}\right) Y = 0 \tag{36}$$

$$\left(\Delta - \frac{\omega}{\alpha_1}\right) Y = 0 \tag{37}$$

Substituting eqn (33) into eqn (36) gives

$$\frac{d^2 Y}{dy^2} + \frac{\beta(n+1)}{1+\beta y}\frac{dY}{dy} + \frac{1}{1+\beta y}\frac{\omega}{\alpha_1} Y = 0 \tag{38}$$

Setting

$$Y = \left(\frac{\xi}{\lambda}\right)^{-n} \Psi \tag{39}$$

$$\xi = \lambda(1+\beta y)^{1/2} \tag{40}$$

$$\lambda^2 = \frac{4\theta}{\alpha_1 \beta^2} \tag{41}$$

and substituting eqns (39)–(41) into eqn (38) gives

$$\frac{d^2\Psi}{d\xi^2} + \frac{1}{\xi}\frac{d\Psi}{d\xi} + \left(1 - \frac{n^2}{\xi^2}\right)\Psi = 0 \tag{42}$$

This is Bessel's equation of n-order. Its general solution can be expressed as

$$\Psi(\xi) = C_1 J_n(\xi) + C_2 Y_n(\xi) \tag{43}$$

Substituting eqn (43) into eqn (39) gives

$$Y(y) = (1+\beta y)^{-n/2}\left[C_1 J_n(\xi) + C_2 Y_n(\xi)\right] \tag{44}$$

Similarly, the general solution of eqn (37) is given by

$$Y(y) = (1+\beta y)^{-n/2}\left[C_1 I_n(\xi) + C_2 K_n(\xi)\right] \tag{45}$$

The sum of eqns (44) and (45) is the general solution of mode shape in the y-axis.

$$Y(y) = (1+\beta y)^{-n/2}\left[C_1 I_n(\xi) + C_2 Y_n(\xi) + C_3 I_n(\xi) + C_4 K_n(\xi)\right] \tag{46}$$

where J_n, Y_n, I_n, K_n are Bessel functions of the first, second, third and fourth kind, respectively.

In order to simplify the calculation, four fundamental functions are introduced herein to express $Y(y)$ as

$$Y(y) = A_1 Y_1(y) + A_2 Y_2(y) + A_3 Y_3(y) + A_4 Y_4(y) \tag{47}$$

The four fundamental functions meet the following conditions:

$$\text{when } y = 0, \quad Y_1 = 1, \quad \frac{dY_1(y)}{dy} = 0 \tag{48}$$

$$Y_2 = 0, \quad \frac{dY_2(y)}{dy} = 1 \tag{49}$$

$$Y_3 = 0, \quad \frac{\mathrm{d}Y_3(y)}{\mathrm{d}y} = 0 \tag{50}$$

$$Y_4 = 0, \frac{\mathrm{d}Y_4(y)}{\mathrm{d}y} = 0 \tag{51}$$

It should be pointed out that the higher derivative of $Y_3(y)$ is different from that of $Y_4(y)$. The procedure of determining the four fundamental functions is as follows:

1. The fundamental functions are expressed in the linear combination of the four special solutions as follows

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \left(\frac{\xi}{\lambda}\right)^{-n} & J_n(\xi) \\ \left(\frac{\xi}{\lambda}\right)^{-n} & Y_n(\xi) \\ \left(\frac{\xi}{\lambda}\right)^{-n} & I_n(\xi) \\ \left(\frac{\xi}{\lambda}\right)^{-n} & K_n(\xi) \end{bmatrix} \tag{52}$$

2. Determining the constants $a_{ij}\,(i, j = 1, 2, 3, 4)$ and the fundamental functions.

The constants $a_{ij}\,(i, j = 1, 2, 3, 4)$ and the fundamental functions can be determined based on the following equations:

$$\begin{bmatrix} \left(\frac{\xi}{\lambda}\right)^{-n} J_n(\xi) & \left(\frac{\xi}{\lambda}\right)^{-n} Y_n(\xi) \\ \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(\frac{\xi}{\lambda}\right)^{-n} J_n(\xi)\right] & \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(\frac{\xi}{\lambda}\right)^{-n} Y_n(\xi)\right] \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \tag{53}$$

$$\begin{bmatrix} \left(\frac{\xi}{\lambda}\right)^{-n} I_n(\xi) & \left(\frac{\xi}{\lambda}\right)^{-n} K_n(\xi) \\ \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(\frac{\xi}{\lambda}\right)^{-n} I_n(\xi)\right] & \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(\frac{\xi}{\lambda}\right)^{-n} K_n(\xi)\right] \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{14} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \tag{54}$$

$$\begin{bmatrix} \left(\frac{\xi}{\lambda}\right)^{-n} J_n(\xi) & \left(\frac{\xi}{\lambda}\right)^{-n} Y_n(\xi) \\ \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(\frac{\xi}{\lambda}\right)^{-n} J_n(\xi)\right] & \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(\frac{\xi}{\lambda}\right)^{-n} Y_n(\xi)\right] \end{bmatrix} \begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \tag{55}$$

$$\begin{bmatrix} \left(\frac{\xi}{\lambda}\right)^{-n} I_n(\xi) & \left(\frac{\xi}{\lambda}\right)^{-n} K_n(\xi) \\ \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(\frac{\xi}{\lambda}\right)^{-n} I_n(\xi)\right] & \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(\frac{\xi}{\lambda}\right)^{-n} K_n(\xi)\right] \end{bmatrix} \begin{bmatrix} a_{23} \\ a_{24} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \tag{56}$$

$$\begin{bmatrix} \left(\frac{\xi}{\lambda}\right)^{-n} J_n(\xi) & \left(\frac{\xi}{\lambda}\right)^{-n} Y_n(\xi) \\ \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(\frac{\xi}{\lambda}\right)^{-n} J_n(\xi)\right] & \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(\frac{\xi}{\lambda}\right)^{-n} Y_n(\xi)\right] \end{bmatrix} \begin{bmatrix} a_{31} \\ a_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \tag{57}$$

$$\begin{bmatrix} \left(\frac{\xi}{\lambda}\right)^{-n} I_n(\xi) & \left(\frac{\xi}{\lambda}\right)^{-n} K_n(\xi) \\ \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(\frac{\xi}{\lambda}\right)^{-n} I_n(\xi)\right] & \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(\frac{\xi}{\lambda}\right)^{-n} K_n(\xi)\right] \end{bmatrix} \begin{bmatrix} a_{33} \\ a_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} \tag{58}$$

$$\begin{bmatrix} \left(\frac{\xi}{\lambda}\right)^{-n} J_n(\xi) & \left(\frac{\xi}{\lambda}\right)^{-n} Y_n(\xi) \\ \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(\frac{\xi}{\lambda}\right)^{-n} J_n(\xi)\right] & \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(\frac{\xi}{\lambda}\right)^{-n} Y_n(\xi)\right] \end{bmatrix} \begin{bmatrix} a_{41} \\ a_{42} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \tag{59}$$

$$\begin{bmatrix} \left(\frac{\xi}{\lambda}\right)^{-n} I_n(\xi) & \left(\frac{\xi}{\lambda}\right)^{-n} K_n(\xi) \\ \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(\frac{\xi}{\lambda}\right)^{-n} I_n(\xi)\right] & \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(\frac{\xi}{\lambda}\right)^{-n} K_n(\xi)\right] \end{bmatrix} \begin{bmatrix} a_{43} \\ a_{44} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} \tag{60}$$

Solving eqns (53)–(60) obtains $a_{ij}\,(i, j = 1, 2, 3, 4)$, then, substituting a_{ij} into eqn (52) gives the four fundamental functions as follows

$$Y_1 = \frac{\pi\lambda}{4}\left(\frac{\xi}{\lambda}\right)^{-n}\left\{ -Y_{n+1}(\lambda)J_n(\xi) + Y_n(\xi)J_{n+1}(\lambda) + \frac{2}{\pi}\left[K_{n+1}(\lambda)I_n(\xi) + K_n(\xi)I_{n+1}(\lambda)\right]\right\} \tag{61}$$

$$Y_2 = \frac{\pi}{2\beta}\left(\frac{\xi}{\lambda}\right)^{-n}\left\{ -Y_n(\lambda)J_n(\xi) + Y_n(\xi)J_n(\lambda) + \frac{2}{\pi}\left[K_n(\lambda)I_n(\xi) - K_n(\xi)I_n(\lambda)\right]\right\} \tag{62}$$

$$Y_3 = \frac{\pi}{2\beta}\left(\frac{\xi}{\lambda}\right)^{-n}\left\{ -Y_n(\lambda)J_n(\xi) + Y_n(\xi)J_n(\lambda) - \frac{2}{\pi}\left[K_n(\lambda)I_n(\xi) - K_n(\xi)I_n(\lambda)\right]\right\} \tag{63}$$

$$Y_4 = \frac{\pi\lambda}{4}\left(\frac{\xi}{\lambda}\right)^{-n}\left\{-Y_{n+1}(\lambda)J_n(\xi) + Y_n(\xi)J_{n+1}(\lambda) - \frac{2}{\pi}\left[K_{n+1}(\lambda)I_n(\xi) + K_n(\xi)I_{n+1}(\lambda)\right]\right\} \tag{64}$$

Case 3:

$$K_1(y) = K_1(1+\beta y)^{n+4} \tag{65}$$

$$\bar{m}_{xy} = \bar{m}(1+\beta y)^n \tag{66}$$

Substituting eqns (65) and (66) into eqn (31) gives

$$\xi^4\frac{\mathrm{d}^4Y}{\mathrm{d}\xi^4} + 2(n+4)\xi^3\frac{\mathrm{d}K_y}{\mathrm{d}y}\frac{\mathrm{d}^3Y}{\mathrm{d}y^3} + (n+4)(n+3)\xi^2\frac{\mathrm{d}^2Y}{\mathrm{d}\xi^2} - \omega_d^4 = 0 \tag{67}$$

where

$$\xi = 1+\beta y, \quad \omega_d^4 = \frac{\omega^2}{\alpha_1^2\beta^4} \tag{68}$$

eqn (67) is Euler's equation.
Letting

$$\xi = \exp(\eta) \tag{69}$$

Substituting eqn (69) into eqn (67) gives

$$\left[D(D-1)(D-2)(D-3) + 2(n+4)D(D-1)(D-2) + (n+4)(n+3)D(D-1) - \omega_d^4\right]Y = 0 \tag{70}$$

where

$$D = \frac{\mathrm{d}}{\mathrm{d}\eta} \tag{71}$$

eqn (70) can be simplified to

$$[D^4 + 2(n+1)D^3 + \left(n^2+n-1\right)D^2 - (n+1)(n+2)D - \omega_d^4]Y = 0 \tag{72}$$

Now we identify $D^iY = r^i$ and then eqn (72) can be expressed as

$$r^4 + 2(n+1)r^3 + \left(n^2+n-1\right)r^2 - (n+1)(n+2)r - \omega_d^4 = 0 \tag{73}$$

The above equation can be rewritten as

$$\left[r^2 + (n+1)r + \frac{n+2}{2}\right]^2 - \omega_d^4 + \left(\frac{n+2}{2}\right)^2 = 0 \tag{74}$$

Solving this equation for r gives

$$r_{1,2} = -\frac{1}{2}\left[n+1 \pm \sqrt{n^2+3-\sqrt{\left(\frac{n+2}{2}\right)^2 + \omega_d^4}}\right] \tag{75}$$

$$r_{3,4} = -\frac{1}{2}\left[n+1 \pm \sqrt{n^2 - 3 - \sqrt{\left(\frac{n+2}{2}\right)^2 + \omega_d^4}}\right] \tag{76}$$

It is easily found that r_1 and r_2 are real roots. If r_3 and r_4 are also real roots, then the general solution of eqn (31) is given by

$$Y = C_1 \exp(r_1\eta) + C_2 \exp(r_2\eta) + C_3 \exp(r_3\eta) + C_4 \exp(r_4\eta) \tag{77}$$

If r_3 and r_4 are complex numbers, then

$$Y = C_1 \exp(r_1\eta) + C_2 \exp(r_2\eta) + \exp\left(-\frac{n+1}{2}\eta\right)(C_3 \cos \varepsilon\eta + C_4 \sin \varepsilon\eta) \tag{78}$$

in which

$$\varepsilon = \sqrt{\sqrt{\left(\frac{n+2}{2}\right)^2 + \omega_d^4} - n^2 + 3} \tag{79}$$

Case 4:

$$K_1(y) = K_1 \exp(-\beta y) \tag{80}$$

$$\bar{m}_{xy} = \bar{m} \exp(-\beta y) \tag{81}$$

Substituting eqns (80) and (81) into eqn (31) leads to a differential equation with constant coefficients as follows

$$K_1 \frac{d^4 Y}{dy^4} - 2K_1 b \frac{d^3 Y}{dy^3} + K_1 b^2 \frac{d^2 Y}{dy^2} - \bar{m}\omega^2 Y = 0 \tag{82}$$

We identify $(d^3 Y/dy^3) = r^3$, $(d^2 Y/dy^2) = r^2$, $(dY/dy) = r$ and then eqn (82) can be expressed as

$$[r(r-b) + \omega_e^2][r(r-b) - \omega_e^2] = 0 \tag{83}$$

where

$$\omega_e^4 = \frac{\omega^2}{\alpha_1^2} \tag{84}$$

The roots of eqn (83) are found as

$$r_{1,2} = \frac{b^2}{4} \pm \sqrt{\frac{b^2}{4} + \omega_e^2} \tag{85}$$

$$r_{3,4} = \frac{b}{2} \pm \sqrt{\frac{b^2}{4} + \omega_e^2} \tag{86}$$

It is obvious that $r_{1,2}$ are real roots and $r_{3,4}$ are complex roots if $\omega_e^2 > (b^2/4)$. Thus, the general solution of eqn (31) can be expressed as

$$Y = \exp\left(\frac{by}{2}\right)\left[C_1 \exp\left(\sqrt{\frac{b^2}{4}+\omega_e^2 y}\right) + C_2 \exp\left(-\sqrt{\frac{b^2}{4}-\omega_e^2 y}\right) + C_3 \sin\left(\sqrt{\omega_e^2+\frac{b^2}{4} y}\right)\right.$$
$$\left. + C_4 \cos\left(-\sqrt{\omega_e^2-\frac{b^2}{4} y}\right)\right] \tag{87}$$

The general solutions of the above four cases can be expressed in a unified formula in terms of the four special solutions as follows

$$Y(y) = C_1 W_1(y) + C_2 W_2(y) + C_3 W_3(y) + C_4 W_4(y) \tag{88}$$

in which $W_j(y)$ ($j = 1, 2, 3, 4$) can be found from eqn (28) for Case 1, from eqns (61)–(64) for Case 2, from eqn (77) or eqn (78) for Case 3, from eqn (87) for Case 4, respectively. C_j ($j = 1, 2, 3, 4$) are constants, which can be determined according to the boundary conditions.

The natural frequencies and mode shapes of cantilever flexural-shear plate can be found in terms of the following procedures:

1. The relationship of parameters at the top and those at the base can be expressed as

$$\left[Y(H) \quad \frac{\mathrm{d}Y(H)}{\mathrm{d}y} \quad 0 \quad 0\right]^{\mathrm{T}} = [T][0 \quad 0 \quad M_y(0) \quad Q_y(0)]^{\mathrm{T}} \tag{89}$$

in which

$$[T] = [W(H)][W(0)]^{-1} \tag{90}$$

$$[W(y)] = \begin{bmatrix} W_1(y) & W_2(y) & W_3(y) & W_4(y) \\ \frac{\mathrm{d}W_1(y)}{\mathrm{d}y} & \frac{\mathrm{d}W_2(y)}{\mathrm{d}y} & \frac{\mathrm{d}W_3(y)}{\mathrm{d}y} & \frac{\mathrm{d}W_4(y)}{\mathrm{d}y} \\ K_y\frac{\mathrm{d}^2W_1(y)}{\mathrm{d}y^2} & K_y\frac{\mathrm{d}^2W_2(y)}{\mathrm{d}y^2} & K_y\frac{\mathrm{d}^2W_3(y)}{\mathrm{d}y^2} & K_y\frac{\mathrm{d}^2W_4(y)}{\mathrm{d}y^2} \\ -\frac{\mathrm{d}}{\mathrm{d}y}\left[K_y\frac{\mathrm{d}^2W_1(y)}{\mathrm{d}y^2}\right] & -\frac{\mathrm{d}}{\mathrm{d}y}\left[K_y\frac{\mathrm{d}^2W_2(y)}{\mathrm{d}y^2}\right] & -\frac{\mathrm{d}}{\mathrm{d}y}\left[K_y\frac{\mathrm{d}^2W_3(y)}{\mathrm{d}y^2}\right] & -\frac{\mathrm{d}}{\mathrm{d}y}\left[K_y\frac{\mathrm{d}^2W_4(y)}{\mathrm{d}y^2}\right] \end{bmatrix} \tag{91}$$

2. Determination of frequency equation

Using eqn (89) obtains

$$\begin{bmatrix} T_{33} & T_{34} \\ T_{43} & T_{44} \end{bmatrix}\begin{bmatrix} M_y(0) \\ Q_y(0) \end{bmatrix} = 0 \tag{92}$$

in which T_{ij} is the element of $[T]$.

Because $M_y(0) \neq 0$, $Q_y(0) \neq 0$, the frequency equation is

$$T_{33}T_{44} - T_{34}T_{43} = 0 \tag{93}$$

Solving eqn (93) obtains θj $(j = 1, 2, \ldots)$ The jk-th circular natural frequency can be determined from eqn (14).

3. Determination of mode shapes

Substituting θ_j into eqn (92) and setting $Q_{yj}(0) = 1$ (or any value) obtain $M_{yj}(0)$.

4. Determination of the integral constants C_{ij} $(i = 1, 2, 3, 4)$

$$\begin{bmatrix} C_{1j} \\ C_{2j} \\ C_{3j} \\ C_{4j} \end{bmatrix} = [W(0)]^{-1} \begin{bmatrix} 0 \\ 0 \\ M_{yj}(0) \\ Q_{yj}(0) \end{bmatrix} \tag{94}$$

Substituting the calculated C_{ij} into eqn (88) gives the j-th mode shape in the y-axis, $Y_j(y)$. The jk-th mode shape can be determined by using eqn (10).

4. Numerical example 1

A residence building that is located in Beijing, China, has 15 storeys with shear-wall structures as shown in Fig. 3. The distribution of shear-wall along the longitudinal direction of the building is uniform and the cross-sectional dimensions of the shear-walls are the same. The foundation of the building can be treated as firm. Thus, the building is simplified as a uniform cantilever flexural-shear plate for free vibration analysis. The procedure for determining the natural frequencies and mode shape of the building is as follows:

1. Determination of mass intensity per unit area of the flexural-shear plate

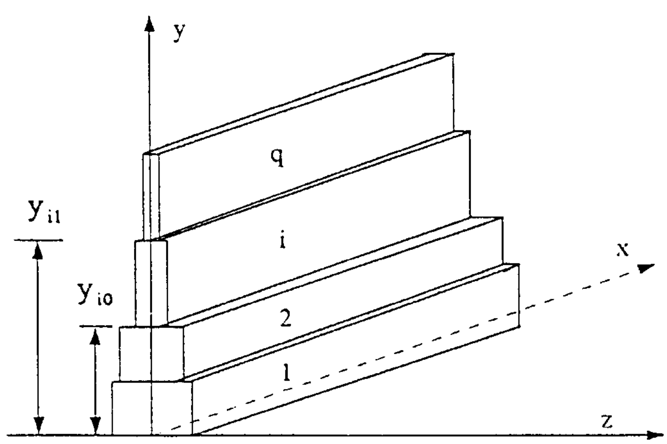

Fig. 2. A multi-step flexural-shear plate.

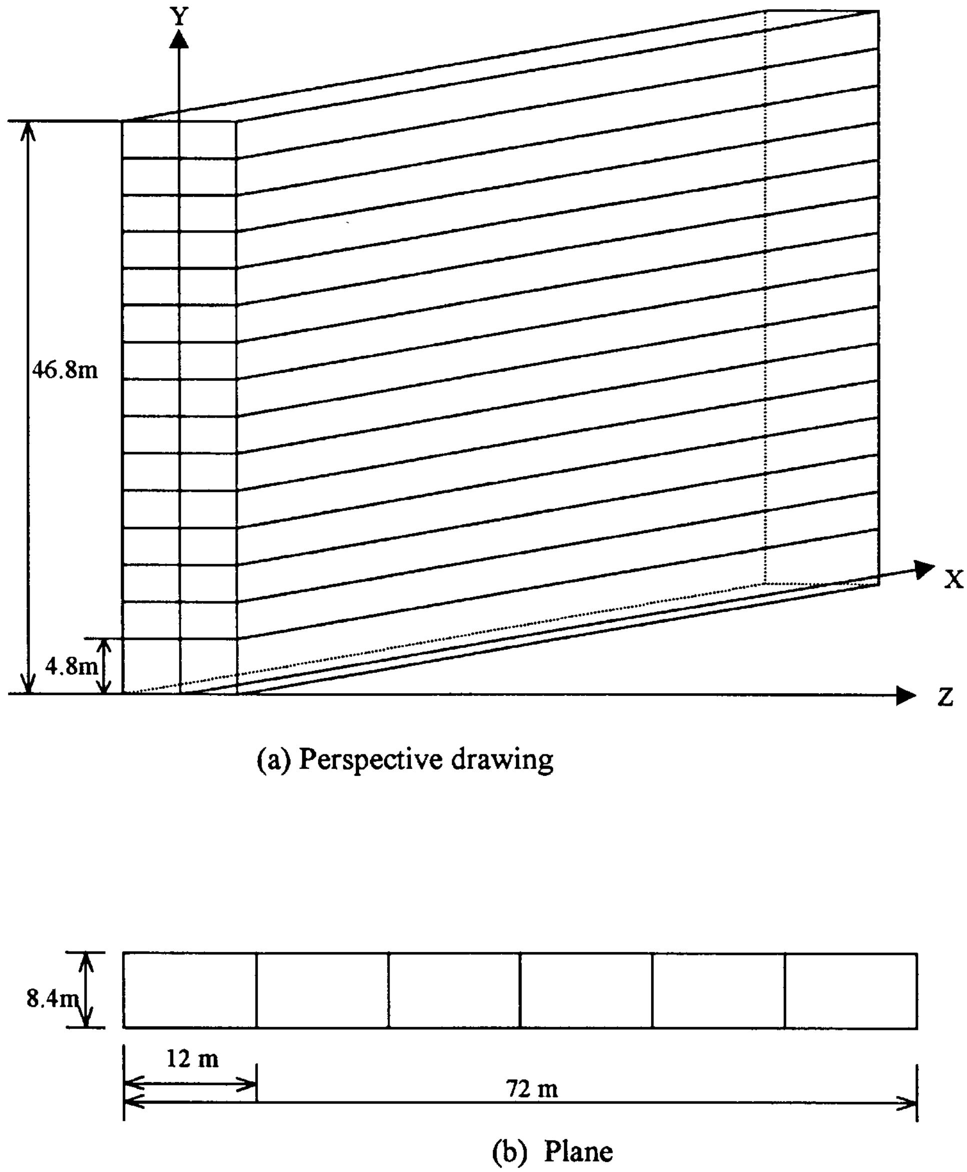

Fig. 3. A narrow building.

The mass distribution is uniform and its intensity per unit area is found as follows

$$\bar{m} = \frac{M}{HL} = \frac{7{,}413{,}120\ \text{kg}}{46.80\ \text{m} \times 72.00\ \text{m}} = 2200\ \text{kg/m}^2$$

where M, H and L are the total mass, the height and the length of the building, respectively (see Fig. 3).

2. Determination of the flexural stiffness, K_1

The total moment of inertia of shear-walls is

$$I = 1.15 \times 10^2 \ \text{m}^4$$

The Young's modulus is

$$E = 2.55 \times 10^{10} \ \text{N/m}^2$$

The total flexural stiffness of shear-walls is

$$EI = 2.93 \times 10^{12} \ \text{N} \cdot \text{m}^2$$

The flexural stiffness, K_1, of the plate is the value of EI divided by the length of the plate (i.e., the length of the building),

$$K_1 = \frac{EI}{L} = \frac{2.93 \times 10^{12}}{72} = 4.07 \times 10^{10} \ \text{N} \cdot \text{m}$$

3. Determination of the shear stiffness, K_2, in the x-axis

The shear stiffness, K_2, in the x-axis is the stiffness of each floor, GF, divided by the storey height, which is found as

$$K_2 = 11.43 \times 10^8 \ \text{N/m}$$

4. Determination of natural frequencies and mode shapes

The k-th circular natural frequency can be found from eqn (23) as

$$\Omega_k = \frac{(k \times 1)\pi}{L}\sqrt{\frac{K_2}{\bar{m}}} = 31.45(k - 1)$$

Letting $k = 1, 2, 3$, obtains $\Omega_1 = 0, \Omega_2 = 31.45, \Omega_3 = 62.90$.
θ_j can be determined from the eigenvalue equation, eqn (93), which is given by

$$ch\bar{k}H \cdot \cos\bar{k}H + 1 = 0$$

It is found that $\theta_1 = 6.90, \theta_2 = 43.27, \theta_3 = 120.97$

The circular natural frequencies, ω_{jk}, can be determined from eqn (14) and the calculated values of ω_{jk} are listed in Table 1. It is necessary to point out that ω_{jk} is corresponding to the j-th mode shape in the y-axis and the k-th mode shape in the x-axis.

It is obvious that $\theta_j = \omega_{jk}$ if K_2 is infinity, i.e., the stiffness of each floor can be treated as infinitely rigid in-plane of the floor. In this case, the vibration modes corresponding to Ω_k $(k > 2)$ may not appear in the vibration of this building. Thus, these types of buildings can be simplified as a flexural bar for vibration analysis.

The mode shapes, $Y_j\,(y)\,(j = 1, 2, 3)$ in the y-axis and $X_k(x)$, $(k = 1, 2, 3)$ in the x-axis are calculated

Table 1
The circular natural frequencies of the narrow building

ω_{11}	ω_{12}	ω_{21}	ω_{13}	ω_{22}	ω_{23}	ω_{31}	ω_{32}	ω_{33}
6.90	32.20	43.27	53.49	63.28	76.05	120.97	124.99	136.35

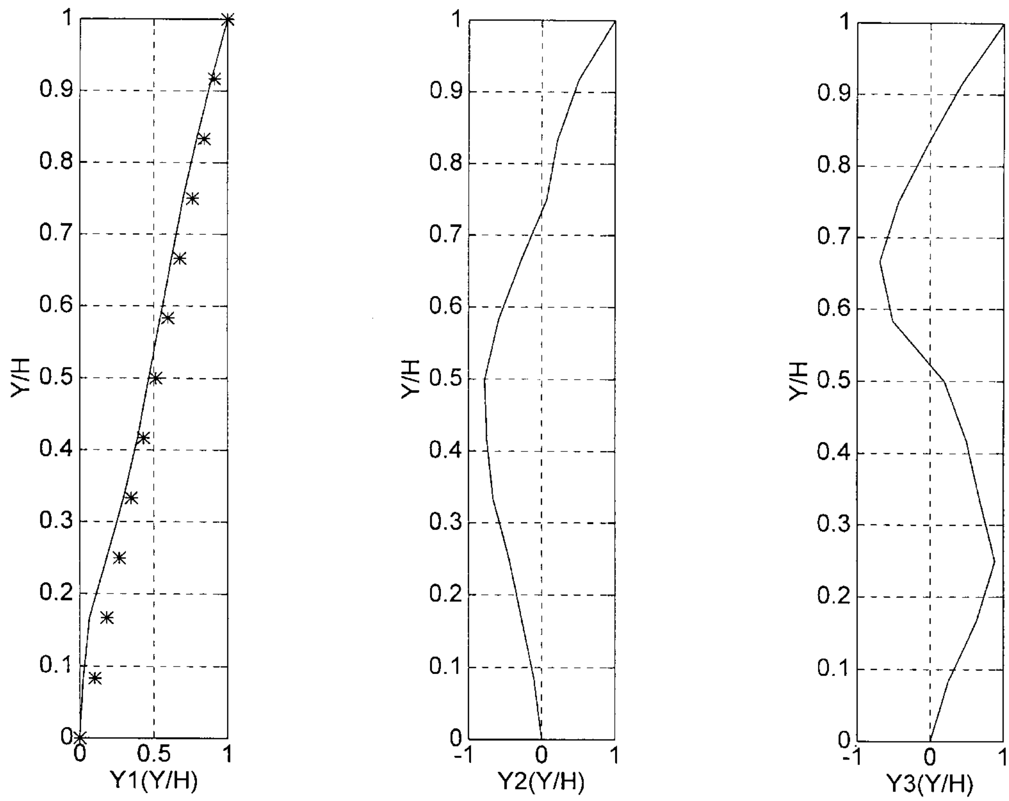

Fig. 4. The first three mode shapes in the y-direction. Note: the solid lines represent the calculated mode shapes and the cross symbols are the measured results by Li et al. (1994).

and shown in Figs. 4 and 5, respectively. The mode shape, $Y_1(y)$ experimentally measured from an ambient vibration survey by Li et al. (1994) are also presented in Fig. 4 for comparison purposes. The measured value of ω_{11} is 6.97. It is clear that the calculated results are in good agreement with the experimental data. This illustrates that the proposed methods in this paper are applicable to engineering application.

The mode shape function, $Z_{jk}(x, y)$, of this building can be found as

$$Z_{jk}(x, y) = Y_j(y)X_k(x)$$

Fig. 6 shows the obtained mode shapes, $Z_{11}(x, y)$, $Z_{22}(x, y)$, $Z_{33}(x, y)$.

5. Numerical example 2

The main structure of the Guangzhou Hotel Building is a R. C. shear-wall structure with 24 stories. There is a three-storey appendage that is built on the top of the main structure. Based on the full-scale measurement of free vibration of this building (Li et al., 1996), the building can be treated as a cantilever flexural-shear plate with variable cross-section in free vibration analysis and the effect of rotatory inertia and transverse shear deformation can be neglected. The procedure for determining the dynamic characteristics of this tall building is as follows:

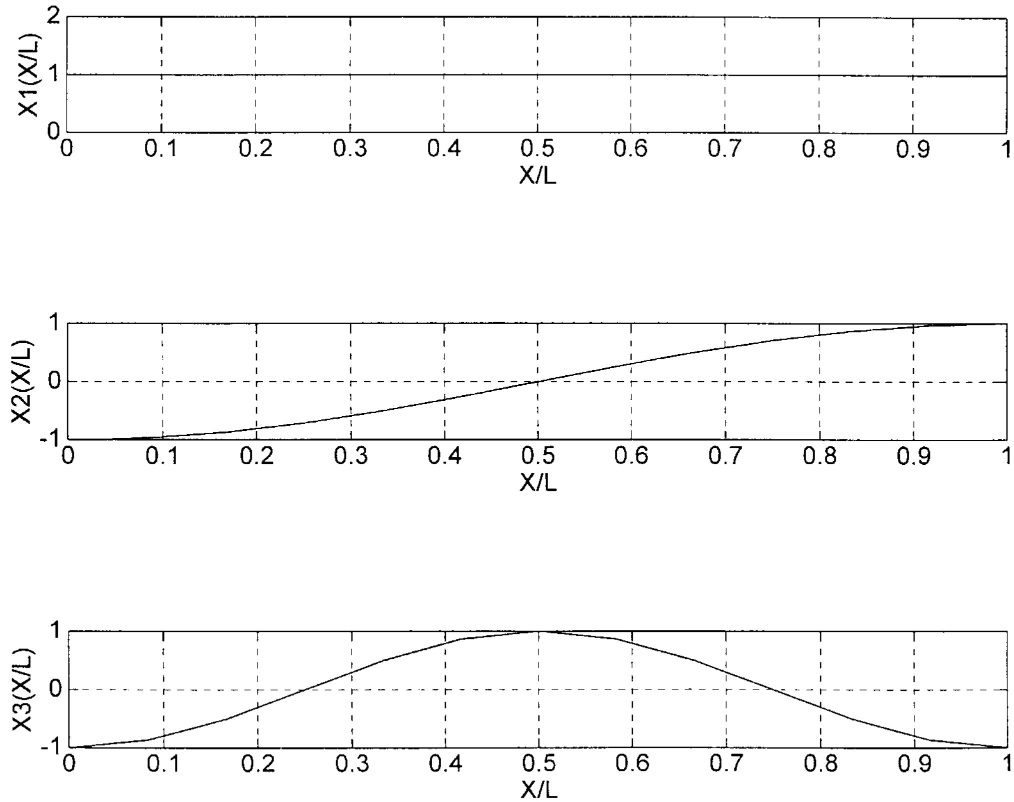

Fig. 5. The first three mode shapes in the x-direction.

1. Determination of the mass per unit area of the plate (Fig. 7)

The mass per unit area of the flexural-shear plate, which represents the Guangzhou Hotel Building, varies in echelon along the building height (Fig. 7). It can be seen in Fig. 7 that the variation of the mass per unit area is comparatively small, thus, it is reasonable to assume $\bar{m}$ as a constant, i.e. $\bar{m} = 3761$ kg/m^2.

2. Evaluation of the flexural stiffness, $K_1(y)$ and the shear stiffness, K_2.

The flexural stiffness, $K_1(y)$, varies in echelon along the height of the building (Fig. 8). For simplification, the distributions of flexural stiffness, $K_1(y)$, is described by a power function of y as follows

$$K_1(y) = K_1(1 + \beta y)^2 \tag{95}$$

According to the following information of this building provided by Li et al. (1994):

$$\text{at } y = 0, \quad K_1(y) = 5.50 \times 10^{11} \ \text{N} \cdot \text{m}$$
$$\text{at } y = H, \quad K_1(y) = 2.80 \times 10^{11} \ \text{N} \cdot \text{m}$$

The constants K_1 and β are determined as

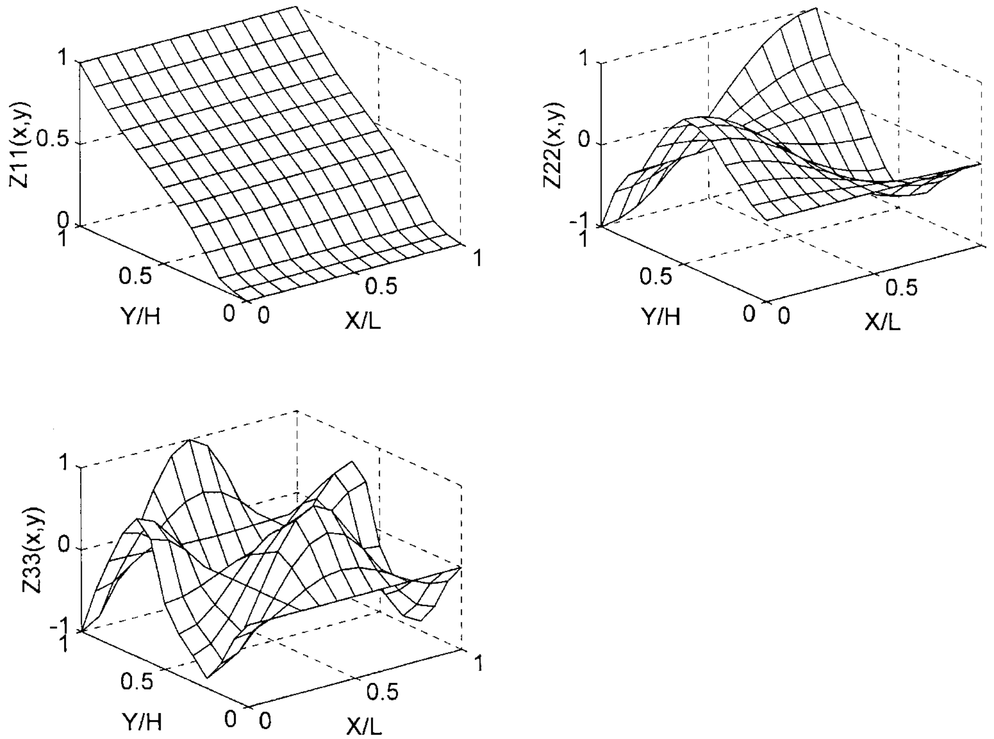

Fig. 6. The mode shapes of the building.

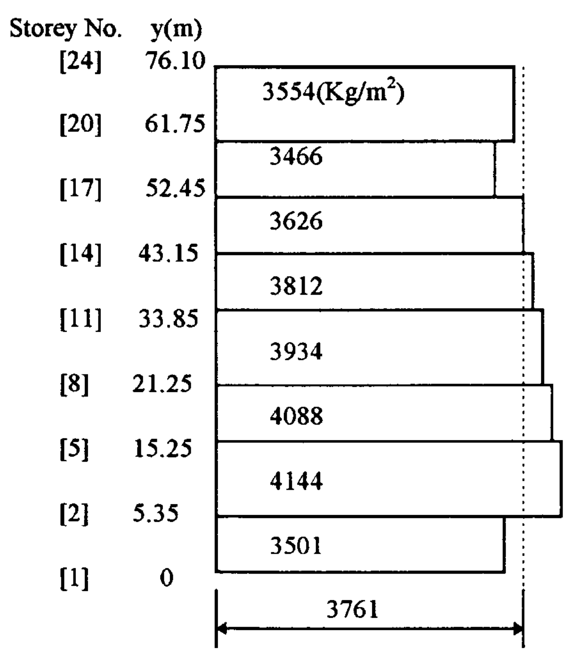

Fig. 7. Mass ($\bar{m}$) distribution of the Guangzhou Hotel Building.

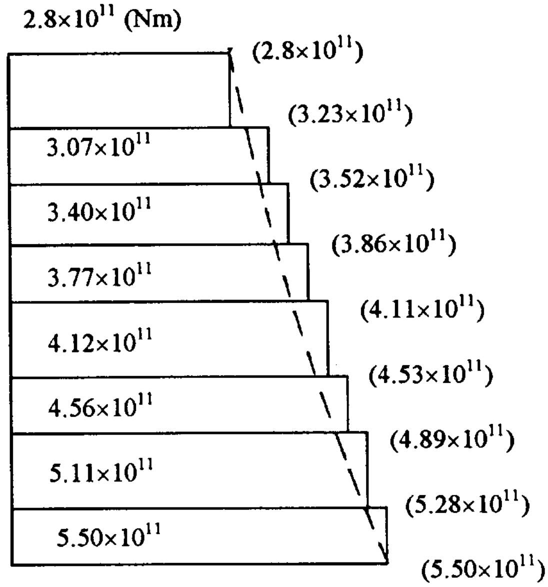

Fig. 8. Stiffness ($K_1(y)$) distribution of the Guangzhou Hotel Building. Note: the dotted lines and values in parentheses are the evaluated distributions.

$$K_1 = 5.50 \times 10^{11} \ \mathrm{N \cdot m}$$
$$\beta = -3.796 \times 10^{-3}$$

The evaluated distribution of stiffness [by eqn (95)] is shown in Fig. 8 (dotted line and the values in parentheses).

The shear stiffness in the x-axis, K_2 is found as

$$K_2 = \frac{GF}{h} = 2.135 \times 10^9$$

where GF is the shear stiffness of floor and h is the storey height.

3. Evaluation of the fundamental natural frequency

As mentioned above, the frequency equation is eqn (93). But, it is more convenient to establish the frequency equation by use of eqn (47). Because the fundamental functions $Y_i(y)$ ($i = 1, 2, 3, 4$) meet eqns (48)–(51), it is easy to determine the integral constants A_i ($i = 1, 2, 3, 4$). According to the boundary conditions, eqn (17) and using eqn (47) ($n = 0$, for this case), we have

$$\text{at } y = 0, \quad Y(0) = 0; \quad \text{gives } A_1 = 0$$

$$\text{at } y = 0, \quad \frac{dY}{dy} = 0, \quad \text{gives } A_2 = 0$$

Then, using the boundary condition, eqn (18), obtains the frequency equations as follows

$$Y_3''(H)Y_4'''(H) = Y_4''(H)Y_3'''(H) \tag{96}$$

Table 2
The fundamental mode shape of the building in the y-axis

Storey level	1	2	5	8	11	14	17	20	24
y/H	0	0.0704	0.2007	0.3230	0.4454	0.5678	0.6976	0.8125	1
$Y_1(y/H)$ measured	0	0.005	0.070	0.160	0.290	0.390	0.540	0.730	1
$Y_1(y/H)$ calculated	0	0.0068	0.0527	0.1414	0.2644	0.4049	0.5599	0.7336	1

where

$$Y_3''(H) = B_1\left\{J_0(\lambda)Y_2(\lambda\bar{H}) - Y_0(\lambda)J_2(\lambda\bar{H}) + \frac{2}{\pi}\left[I_0(\lambda)K_2(\lambda\bar{H}) - K_0(\lambda)I_2(\lambda\bar{H})\right]\right\} \tag{97}$$

$$Y_3'''(H) = B_2\left\{J_0(\lambda)Y_3(\lambda\bar{H}) - Y_0(\lambda)J_3(\lambda\bar{H}) + \frac{2}{\pi}\left[I_0(\lambda)K_3(\lambda\bar{H}) - K_0(\lambda)I_3(\lambda\bar{H})\right]\right\} \tag{98}$$

$$Y_4''(H) = B_3\left\{J_1(\lambda)Y_2(\lambda\bar{H}) - Y_1(\lambda)J_2(\lambda\bar{H}) + \frac{2}{\pi}\left[I_1(\lambda)K_2(\lambda\bar{H}) - K_1(\lambda)I_2(\lambda\bar{H})\right]\right\} \tag{99}$$

$$Y_4'''(H) = B_4\left\{J_1(\lambda)Y_3(\lambda\bar{H}) - Y_1(\lambda)J_3(\lambda\bar{H}) + \frac{2}{\pi}\left[I_1(\lambda)K_3(\lambda\bar{H}) - K_1(\lambda)I_3(\lambda\bar{H})\right]\right\} \tag{100}$$

$$\left.\begin{array}{ll} B_1 = \dfrac{\pi\lambda^2\beta}{8}\bar{H}^{-2} & B_2 = \dfrac{\pi\lambda^3\beta^2}{16\bar{H}}\bar{H}^{-2} \\ B_3 = \dfrac{\pi\lambda^3\beta^2}{16}\bar{H}^{-2} & B_4 = \dfrac{\pi\lambda^4\beta^2}{32\bar{H}}\bar{H}^{-2} \\ & \bar{H} = (1+\beta H)^{1/2} \end{array}\right\} \tag{101}$$

It can be assumed that $B_1 = B_2 = B_3 = B_4 = 1$ for solving the frequency equation. It is found that $\theta_1 = 6.85$ rad/s.

Ω_k can be determined from eqn (23). Ω_1 is found as: $\Omega_1 = 0$.

From eqn (14), it is obvious that $\omega_{11} = \theta_1 = 6.858$ rad/s. The measured value of ω_{11} by Li et al. (1996) is 6.478 rad/s. The computed fundamental natural frequency of this building approaches the measured data, suggesting that the methods proposed herein are applicable to engineering application and practice.

4. Calculation of vibration mode shape

After computing θ_1, the first mode shape in the y-axis, $Y_1(y)$, can be determined by

$$Y(y) = A_3Y_3 + A_4Y_4 \tag{102}$$

If we set $A_3 = 1$, then

$$A_4 = -\frac{Y_3''(H)}{Y_4''(H)} \tag{103}$$

The calculated results are listed in Table 2. The corresponding measured data by Li et al. (1996) are also presented in Table 2 for the purposes of comparison. It can be seen from Table 2 that the calculated fundamental mode shape in the y-axis show good agreement with the measured mode shape.

The first mode shape in the x-axis, $X_1(x)$, can be determined by use of eqn (24) as: $X_1(x) = C$. C can be taken as any value, for example, $C = 1$.

Using the aforementioned procedure, the higher natural frequencies and corresponding mode shapes of this tall building can also be determined.

6. Conclusions

In fact, there are very few equations of vibrating plates with varying cross-section where exact solutions can be obtained. In this paper, an efficient approach to determine the natural frequencies and mode shapes of cantilever flexural-shear plates with variably distributed mass and stiffness has been proposed. The exact solutions for free vibration of cantilever flexural-shear plates are derived by selecting suitable expressions, such as power functions and exponential functions, for the distributions of stiffness and mass along the height of the plates.

The numerical examples demonstrate that the calculated natural frequencies and mode shapes of two narrow buildings are in good agreement with the corresponding full-scale measurements. It is shown through the numerical examples that the selected expressions are suitable for describing the distributions of mass and stiffness of typical multi-storey buildings. It was found that if the stiffness of each floor of a narrow building can be treated as infinitely rigid, then, the mode shapes of a flexural-shear plate which represents the narrow building are the same as those of a flexural bar which is a special case of a flexural-shear plate. Thus, the methods proposed in this paper are suitable for the calculation of free vibration of narrow buildings and common shear-wall buildings.

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